ADJUNCTION OF ROOTS TO UNITRIANGULAR GROUPS OVER PRIME FINITE FIELDS

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Abstract. In this paper we study embeddings of unitriangular groups $\mathrm{UT}_n(\mathbb{F}_p)$ arising under adjunction of roots. We construct embeddings of $\mathrm{UT}_n(\mathbb{F}_p)$ in $\mathrm{UT}_m(\mathbb{F}_p)$, for $n \geq 2$, $m = (n-1)p^s+1$, $s \in \mathbb{Z}^+$, such that any element of $\mathrm{UT}_n(\mathbb{F}_p)$ has a p^s-th root in $\mathrm{UT}_m(\mathbb{F}_p)$. Also we construct an embedding of the wreath product $\mathrm{UT}_n(\mathbb{F}_p) \wr C_{p^s}$ in $\mathrm{UT}_m(\mathbb{F}_p)$, where C_{p^s} is the cyclic group of order p^s.

1. Introduction

Equations over groups is old and well-established area of the group theory. B. H. Neumann started its systematic investigation in [7]. We refer to the survey [9] for developments in this area. Also a brief historical note could be found in [4].

An equation with the variable x over a group G is an expression of the form

$$w(x) = 1, \tag{1.1}$$

where $w(x) \in G * \langle x \rangle$ is a group word formed with x and elements of G.

If H is a bigger group, i.e., a group containing G as a fixed subgroup, then an equation over G could be also considered as an equation over H. Equation (1.1) is *solvable in* G if there is an element $g \in G$ such that $w(g) = 1$. Equation (1.1) is *solvable over* G if there is an overgroup $H \geq G$ where this equation has a solution. In the latter case we may assume that H is generated by a solution of (1.1) and elements of G. In other words, H is obtained by *adjoining* a solution of (1.1) to G. We may also call H an *extension* of G.

In [7] B. H. Neumann studied conditions on G and $w(x)$ under which equation (1.1) is solvable over G and obtained solution of this problem in general case [7, Theorem 2.3]. Also he proved the following:

Theorem 1.1 (B. H. Neumann, [7]). *The equation*

$$x^m = g \tag{1.2}$$

is solvable over an arbitrary group G for any $g \in G$ and $m \in \mathbb{Z}^+$.

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Solution of (1.2) is called *m-th root* of g. To prove Theorem 1.1 B. H. Neumann used amalgamated free products. An alternative way using wreath products was suggested by G. Baumslag in [2].

From Theorem 1.1 we obtain the following result:

Theorem 1.2 (B. H. Neumann, [7]). *Every group G is isomorphic to a subgroup of a group D in which every element has an n-th root for every $n \in \mathbb{Z}^+$.*

Such a group D is called a *divisible* group.

According to [12], in 1960 B. H. Neumann posed the following problem: given a nilpotent group B, an element h of B and a positive integer n, is it always possible to embed B in a nilpotent group containing an n-th root for h? In other words, whether *nilpotent* adjunction of n-th root to h is possible? This problem was studied in [12, 1]. In particular J. Wiegold in [12] proved that it is always possible to nilpotently adjoint n-th roots to elements of finite order. We remark that the answer to the problem posed by B. H. Neumann is, in general, negative. For suppose that p is any given prime. Then there exists a nilpotent group G (infinitely generated) with an element u (of infinite order) such that any overgroup H of G in which u has a p-th root is not nilpotent [8].

We will mention some results of G. Baumslag [3] related to the problem above.

Theorem 1.3 (G. Baumslag, [3]). *If p is any prime and G any given finitely generated nilpotent group, then G can be embedded in a nilpotent group H so that any element u in G now has a p-th root in H.*

Theorem 1.4 (G. Baumslag, [3]). *Any finitely generated nilpotent group can be embedded in a locally nilpotent group which is divisible.*

We also remark that in [5] A. Mal'cev proved that any torsion free nilpotent group can be embedded in a divisible nilpotent group of the same class.

Let $\mathbb{F}_p$ be the prime finite field of order p and $\mathrm{UT}_n(\mathbb{F}_p)$ ($n \geq 2$) be the group of $n \times n$ upper unitriangular matrices over $\mathbb{F}_p$. In this paper we will be interested in adjunction of p^s-th roots, $s \in \mathbb{Z}^+$, to the group $\mathrm{UT}_n(\mathbb{F}_p)$, such that an overgroup is of the form $\mathrm{UT}_m(\mathbb{F}_p)$ for some $m \geq n$. Observe that we only need to perform adjunction of p^s-th roots to a finite p-group G, since G contains n-th roots for any n such that $\gcd(n, p) = 1$.

In the group $\mathrm{UT}_n(\mathbb{F}_p)$ by $t_{i,j}$ ($1 \leq i < j \leq n$) we denote a *transvection* $e + e_{i,j}$, where e is the identity matrix, $e_{i,j}$ is the matrix having 1 in the ij-component and 0 otherwise. Also for any $\gamma \in \mathbb{F}_p$ we denote $t_{i,j}(\gamma) = e + \gamma e_{i,j}$.

In Lemma 3.1 we perform adjunction of roots to transvections in $\mathrm{UT}_n(\mathbb{F}_p)$. In Theorem 3.4 we perform *simultaneous* adjunction of q-th roots ($q = p^s$, $s \in \mathbb{Z}^+$) to $\mathrm{UT}_n(\mathbb{F}_p)$, i.e., we construct the embeddings of $\mathrm{UT}_n(\mathbb{F}_p)$ in $\mathrm{UT}_m(\mathbb{F}_p)$, where $m = (n-1)q + 1$, such that any element of $\mathrm{UT}_n(\mathbb{F}_p)$ has a q-th root in $\mathrm{UT}_m(\mathbb{F}_p)$.

It is well known that any finite p-group G is isomorphic to a subgroup of $\mathrm{UT}_{|G|}(\mathbb{F}_p)$. Observe that the wreath product $\mathrm{UT}_n(\mathbb{F}_p) \wr C_q$ of the group $\mathrm{UT}_n(\mathbb{F}_p)$ with the cyclic group of order q contains a q-th root for any element of $\mathrm{UT}_n(\mathbb{F}_p)$ and is also a finite p-group. So it embeds in $\mathrm{UT}_m(\mathbb{F}_p)$ for some $m > n$. In Theorem 4.3 we construct the embedding of $\mathrm{UT}_n(\mathbb{F}_p) \wr C_q$ in $\mathrm{UT}_m(\mathbb{F}_p)$, where $m = (n-1)q + 1$. This value of m is the minimal possible, since, according to Lemma 2.4, the nilpotency class of $\mathrm{UT}_n(\mathbb{F}_p) \wr C_q$ is equal to $(n-1)q$. In Lemma 4.4 we show that theorems 3.4 and 4.3 lead to the same result.

2. Preliminaries

We will outline the definition of the wreath product $G \wr C$. Let G^C be the group of all mappings from C to G with multiplication defined by $(f \cdot f')(t) = f(t)f'(t)$, for all $t \in C$. The group G^C is called the base group of the wreath product. The group $G \wr C$ is the set of pairs $\{sf \mid s \in C, f \in G^C\}$ with multiplication

$$sf \cdot s'f' = ss'f^{s'}f',$$

where $f^{s'}(t) = f(ts'^{-1})$, for all $t \in C$.

If $C = C_n(c)$ is the cyclic group of order n, generated by c, then an element f of the base group is essentially a tuple of length n

$$(f(c^0), f(c^1), \ldots, f(c^{n-1})).$$

An element f^c is a tuple

$$(f(c^{n-1}), f(c^0), \ldots, f(c^{n-2})),$$

i.e., the generator c of the cyclic group acts on elements of the base group as the right cyclic shift.

Using wreath products for adjunction of roots, the group G is usually identified with the diagonal subgroup of the base group.

In [2] G. Baumslag proved that the wreath product $A \wr B$ of two nontrivial nilpotent groups is nilpotent if and only if both A and B are p-groups with A of finite exponent and B finite. Since that there were many attempts to find the class of a nilpotent wreath product. It was finally obtained by D. Shield (see [6, 10, 11] for details).

Definition 2.1. *Let G be a group and p be a prime. The K_p-series of G is defined by*

$$K_{i,p}(G) = \prod_{np^j \geq i} \gamma_n(G)^{p^j},$$

where $i \geq 1$ and $\gamma_n(G)$ is the n-th term of the lower central series of G. In particular $K_{1,p}(G) = G$.

The K_p-series of a finite p-group will eventually reach the trivial group. Hence the following definition makes sense.

Definition 2.2. *Let B be a finite p-group, for a prime p. Let d be the maximal integer such that $K_{d,p}(B) \neq \{1\}$. For each v, $v = 1, \dots, d$, define $e(v)$ by*

$$p^{e(v)} = |K_{v,p}(B)/K_{v+1,p}(B)|$$

and define a and b by

$$a = 1 + (p-1)\sum_{v=1}^{d} ve(v),$$
$$b = (p-1)d.$$

We can now state Shield's result.

Theorem 2.3 (D. Shield, [11]). *Let p be a prime, A a p-group, nilpotent of class c, and of finite exponent, and let B be a finite p-group, with a and b defined as in Definition 2.2. Define $s(w)$ by $p^{s(w)}$ is the exponent of $\gamma_w(A)$, for $w = 1, \dots, c$. Then*

$$\mathrm{cl}(A \wr B) = \max_{1 \le w \le c}\{aw + (s(w)-1)b\},$$

where $\mathrm{cl}(G)$ is the nilpotency class of G.

We will apply the theorem above to prove the following lemma.

Lemma 2.4. *Let p be a prime, $q = p^s$, $s \in \mathbb{Z}^+$, and $n \ge 2$, then*

$$\mathrm{cl}(\mathrm{UT}_n(\mathbb{F}_p) \wr C_q) = q(n-1).$$

Proof. First we will compute for the group $G = C_q$ its K_p-series (by Definition 2.1). It is easy to check that K_p-series of G has the form

$$K_{1,p} = G \ge \overbrace{G^p \ge \dots \ge G^p}^{p-1} \ge \overbrace{G^{p^2} \ge \dots \ge G^{p^2}}^{p(p-1)} \ge \dots$$
$$\ge \overbrace{G^{p^{s-1}} \ge \dots \ge G^{p^{s-1}}}^{p^{s-2}(p-1)} \ge K_{p^{s-1}+1,p} = \{1\}.$$

Then, using notation of Definition 2.2, the sequence $e(v)$ ($v = 1, \dots, d$) has the form

$$\overbrace{1, 0, \dots, 0}^{p-1}, \overbrace{1, 0, \dots, 0}^{p(p-1)}, \dots, \overbrace{1, 0, \dots, 0}^{p^{s-2}(p-1)}, 1$$

and $d = p^{s-1}$, $b = (p-1)p^{s-1}$,

$$a = 1 + (p-1)\sum_{v=1}^{d} ve(v) = 1 + (p-1)(1 + p + p^2 + \dots + p^{s-1}) = p^s.$$

By Theorem 2.3 the nilpotency class of the wreath product $\mathrm{UT}_n(\mathbb{F}_p) \wr C_q$ is determined by

$$\mathrm{cl}(\mathrm{UT}_n(\mathbb{F}_p) \wr C_q) = \max_{1 \le w \le n-1}\{p^s w + (s(w)-1)(p-1)p^{s-1}\},$$

where $p^{s(w)}$ is the exponent of $\gamma_w(\mathrm{UT}_n(\mathbb{F}_p))$. Observe that $\gamma_{n-1}(\mathrm{UT}_n(\mathbb{F}_p))$ is the cyclic group of order p, hence $s(n-1) = 1$. It is easy to see that maximum is attained when $w = n-1$, so

$$\mathrm{cl}(\mathrm{UT}_n(\mathbb{F}_p) \wr C_q) = p^s(n-1) = q(n-1).$$

□

3. Adjunction of roots

We will introduce the following notations for the subgroups of $\mathrm{UT}_n(\mathbb{F}_p)$:

$$\begin{aligned} \mathrm{FR}_n &= \{(a_{ij}) \in \mathrm{UT}_n(\mathbb{F}_p) \mid a_{ij} = 0, j > i > 1\}, \\ \mathrm{LC}_n &= \{(a_{ij}) \in \mathrm{UT}_n(\mathbb{F}_p) \mid a_{ij} = 0, n > j > i\}, \\ \mathrm{A}_n &= \{(a_{ij}) \in \mathrm{UT}_n(\mathbb{F}_p) \mid a_{1j} = 0, j > 1\}, \\ \mathrm{B}_n &= \{(a_{ij}) \in \mathrm{UT}_n(\mathbb{F}_p) \mid a_{in} = 0, i < n\}. \end{aligned} \tag{3.1}$$

Observe that subgroups FR_n and LC_n are normal in $\mathrm{UT}_n(\mathbb{F}_p)$ and isomorphic to C_p^{n-1}, where C_p is the cyclic group of order p. Subgroups A_n and B_n are naturally isomorphic to $\mathrm{UT}_{n-1}(\mathbb{F}_p)$. Furthermore

$$\mathrm{UT}_n(\mathbb{F}_p) = \mathrm{FR}_n \leftthreetimes \mathrm{A}_n = \mathrm{LC}_n \leftthreetimes \mathrm{B}_n.$$

Let $t_{i,j}$ denote a transvection in the group $\mathrm{UT}_n(\mathbb{F}_p)$. It is known that $t_{i,j}$ satisfy the following relations

$$\begin{aligned} [t_{i,j}, t_{j,k}] &= t_{i,k}, \\ [t_{i,j}, t_{k,l}] &= 1, \qquad j \neq k,\ i \neq l, \\ t_{i,j}^p &= 1, \end{aligned} \tag{3.2}$$

and any other relation between elements of $\mathrm{UT}_n(\mathbb{F}_p)$ is a consequence of relations (3.2).

Let $m > n$ and $1 = k_1 < k_2 \cdots < k_n = m$ be a sequence of integers. By $t'_{i,j}$ we denote a transvection in $\mathrm{UT}_m(\mathbb{F}_p)$. It is clear that the mapping

$$\phi : t_{i,i+1} \mapsto t'_{k_i,k_{i+1}}, \quad i = 1, \dots, n-1, \tag{3.3}$$

is an embedding of $\mathrm{UT}_n(\mathbb{F}_p)$ in $\mathrm{UT}_m(\mathbb{F}_p)$.

The following lemma utilizes the embedding above to perform adjunction of roots to transvections in $\mathrm{UT}_n(\mathbb{F}_p)$. Further we will use rational numbers to index rows and columns of matrices.

Lemma 3.1. *The equation*

$$x^{p^s r} = t_{i,j}(\gamma), \tag{3.4}$$

over $UT_n(\mathbb{F}_p)$ ($n \geq 2$), where $\gcd(p, r) = 1$, $s \in \mathbb{Z}^+$, is solvable in an overgroup isomorphic to $UT_m(\mathbb{F}_p)$, where $m = n + p^s - 1$.

Proof. For brevity denote $q = p^s$. It is well known that $\mathrm{UT}_n(\mathbb{F}_p)$ is generated by $t_{i,i+1}$, for $i = 1, \dots, n-1$. Insert a sequence of $q-1$ numbers $\alpha_l \in \mathbb{Q} \setminus \mathbb{N}$ between i and j such that

$$i < \alpha_1 < \dots < \alpha_{q-1} < j.$$

Positions of α_l relative to indices $i+1, \dots, j-1$ don't have much effect. For simplicity we assume that $1 < \alpha_l < i+1$, for $l = 1, \dots, q-1$.

Let $\mathrm{UT}_n(\mathbb{F}_p)$ be embedded in $\mathrm{UT}_m(\mathbb{F}_p)$, generated by transvections

$$t'_{1,2}, \dots, t'_{i-1,i}, t'_{i,\alpha_1}, t'_{\alpha_1,\alpha_2}, \dots, t'_{\alpha_{q-1},i+1}, t'_{i+1,i+2}, \dots, t'_{n-1,n},$$

by the mapping

$$\phi : t_{i,i+1} \mapsto t'_{i,i+1}, \quad i = 1, \dots, n-1. \tag{3.5}$$

In $\mathrm{UT}_m(\mathbb{F}_p)$ take the element

$$x = e + \gamma_1 e_{i,\alpha_1} + \dots + \gamma_q e_{\alpha_{q-1},j},$$

where $\gamma_1\gamma_2 \dots \gamma_q = r^{-1}\gamma$ (r^{-1} is the inverse for r modulo p), then

$$x^{qr} = (e + r^{-1}\gamma e_{i,j})^r = e + \gamma e_{i,j} = t_{i,j}(\gamma).$$

Hence x is a solution of (3.4). □

The method above, however, doesn't allow to adjoint a root to any element of $\mathrm{UT}_n(\mathbb{F}_p)$. Now for $q = p^s$, $s \in \mathbb{Z}^+$, we will describe the embeddings of $\mathrm{UT}_n(\mathbb{F}_p)$ in $\mathrm{UT}_m(\mathbb{F}_p)$, where $m = (n-1)q + 1$, that naturally arise in Theorem 3.4.

Let $\alpha_{i,j} \in \mathbb{Q}$ be such that

$$i < \alpha_{i,1} < \dots < \alpha_{i,q-1} < i+1, \qquad i = 1, \dots, n-1,$$

and let $\mathrm{UT}_m(\mathbb{F}_p)$ be generated by

$$t'_{i,\alpha_{i,1}}, t'_{\alpha_{i,1},\alpha_{i,2}}, \dots, t'_{\alpha_{i,q-1},i+1}, \qquad i = 1, \dots, n-1.$$

Consider the embedding $\phi : \mathrm{UT}_n(\mathbb{F}_p) \to \mathrm{UT}_m(\mathbb{F}_p)$ defined by

$$\phi : \quad \begin{array}{rcl} t_{1,2} & \mapsto & t'_{1,2}, \\ t_{2,3} & \mapsto & t'_{2,3} t'_{\alpha_{1,1},\alpha_{2,1}} t'_{\alpha_{1,2},\alpha_{2,2}} \dots t'_{\alpha_{1,q-1},\alpha_{2,q-1}}, \\ t_{3,4} & \mapsto & t'_{3,4} t'_{\alpha_{2,1},\alpha_{3,1}} t'_{\alpha_{2,2},\alpha_{3,2}} \dots t'_{\alpha_{2,q-1},\alpha_{3,q-1}}, \\ & \dots & \\ t_{n-1,n} & \mapsto & t'_{n-1,n} t'_{\alpha_{n-2,1},\alpha_{n-1,1}} t'_{\alpha_{n-2,2},\alpha_{n-1,2}} \dots t'_{\alpha_{n-2,q-1},\alpha_{n-1,q-1}}. \end{array} \tag{3.6}$$

We will prove that (3.6) is really an embedding.

Define the following subgroups in $\mathrm{UT}_m(\mathbb{F}_p)$

$$H_i = \left\langle t'_{\alpha_{1,i},\alpha_{2,i}}, t'_{\alpha_{2,i},\alpha_{3,i}}, \dots, t'_{\alpha_{n-2,i},\alpha_{n-1,i}} \right\rangle, \quad i = 1, \dots, q-1,$$
$$H_q = \left\langle t'_{2,3}, t'_{3,4}, \dots, t'_{n-1,n} \right\rangle.$$

Observe that $H_i \simeq \mathrm{UT}_{n-1}(\mathbb{F}_p)$, for $i = 1, \dots, q$, and for $k \neq l$ subgroups H_k and H_l are commuting element-wise. Denote $H = H_1 \times \dots \times H_q$ and $D(H)$ is the diagonal subgroup of H, then $D(H) \simeq \mathrm{UT}_{n-1}(\mathbb{F}_p)$. Consider a

subgroup W of FR_m, consisting of matrices having nonzero elements only in positions $kq+1$, for $k=0,\dots,n-1$. It is easy to see that $W \simeq \mathrm{FR}_n$. Take a semidirect product $P = W \leftthreetimes D(H)$, where an element $(h_1,\dots,h_q) \in D(H)$ acts on $w \in W$ as a conjugation by h_q. Then $P \simeq \mathrm{FR}_n \leftthreetimes \mathrm{UT}_{n-1}(\mathbb{F}_p) \simeq \mathrm{UT}_n(\mathbb{F}_p)$. The basis of P consist of images $\phi(t_{i,i+1})$, $i=1,\dots,n-1$, specified in (3.6). Hence (3.6) is really an embedding.

Example 3.2. Let $n=p=q=3$, then the image of

$$a = \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{UT}_3(\mathbb{F}_3)$$

under embedding (3.6) is equal to

$$\phi(a) = \begin{pmatrix} 1 & 0 & 0 & a_{12} & 0 & 0 & a_{13} \\ 0 & 1 & 0 & 0 & a_{23} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & a_{23} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & a_{23} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{UT}_7(\mathbb{F}_3)$$

Similarly one can define the embedding $\psi : \mathrm{UT}_n(\mathbb{F}_p) \to \mathrm{UT}_m(\mathbb{F}_p)$ by

$$(3.7)\qquad \psi : \begin{array}{rcl} t_{1,2} & \mapsto & t'_{1,2}t'_{\alpha_{1,1},\alpha_{2,1}}t'_{\alpha_{1,2},\alpha_{2,2}}\cdots t'_{\alpha_{1,q-1},\alpha_{2,q-1}}, \\ t_{2,3} & \mapsto & t'_{2,3}t'_{\alpha_{2,1},\alpha_{3,1}}t'_{\alpha_{2,2},\alpha_{3,2}}\cdots t'_{\alpha_{2,q-1},\alpha_{3,q-1}}, \\ & \cdots & \\ t_{n-2,n-1} & \mapsto & t'_{n-2,n-1}t'_{\alpha_{n-2,1},\alpha_{n-1,1}}t'_{\alpha_{n-2,2},\alpha_{n-1,2}}\cdots t'_{\alpha_{n-2,q-1},\alpha_{n-1,q-1}}, \\ t_{n-1,n} & \mapsto & t'_{n-1,n}. \end{array}$$

In a similar way one can prove that (3.7) is really an embedding.

Example 3.3. Let $n=p=q=3$, then the image of

$$a = \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{UT}_3(\mathbb{F}_3)$$

under embedding (3.7) is equal to

$$\psi(a) = \begin{pmatrix} 1 & 0 & 0 & a_{12} & 0 & 0 & a_{13} \\ 0 & 1 & 0 & 0 & a_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & a_{12} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & a_{23} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{UT}_7(\mathbb{F}_3)$$

Using embeddings (3.6) and (3.7) one can adjoint roots to any elements of $\mathrm{UT}_n(\mathbb{F}_p)$.

Theorem 3.4. *The equation*

$$(3.8)\qquad x^q = a,$$

over $UT_n(\mathbb{F}_p)$ $(n \geq 2)$, where $q = p^s$, $s \in \mathbb{Z}^+$, is solvable in an overgroup isomorphic to $UT_m(\mathbb{F}_p)$, where $m = (n-1)q+1$.

Proof. Let $\alpha_{i,j} \in \mathbb{Q}$ be such that

$$i < \alpha_{i,1} < \cdots < \alpha_{i,q-1} < i+1, \qquad i = 1, \ldots, n-1,$$

and let $\mathrm{UT}_m(\mathbb{F}_p)$ be generated by

$$t'_{i,\alpha_{i,1}}, t'_{\alpha_{i,1},\alpha_{i,2}}, \ldots, t'_{\alpha_{i,q-1},i+1}, \qquad i = 1, \ldots, n-1.$$

Write $a = (a_{i,j})$. Using induction on n we will prove that solution of (3.8) with respect to embedding (3.6) has the form

$$x = x_{n-1}x_{n-2} \ldots x_2 x_1, \tag{3.9}$$

where

$$\begin{aligned} x_1 &= t'_{\alpha_{1,q-1},2} \ldots t'_{\alpha_{1,1},\alpha_{1,2}} t'_{1,\alpha_{1,1}}(a_{1,2}), \\ x_2 &= t'_{\alpha_{2,q-1},3} \ldots t'_{\alpha_{2,1},\alpha_{2,2}} t'_{2,\alpha_{2,1}}(a_{2,3}) t'_{1,\alpha_{2,1}}(a_{1,3}), \\ &\ldots \\ x_{n-1} &= t'_{\alpha_{n-1,q-1},n} \ldots t'_{\alpha_{n-1,1},\alpha_{n-1,2}} t'_{n-1,\alpha_{n-1,1}}(a_{n-1,n}) \cdot \\ &\quad \cdot t'_{n-2,\alpha_{n-1,1}}(a_{n-2,n}) \ldots t'_{1,\alpha_{n-1,1}}(a_{1,n}). \end{aligned}$$

Induction base. If $n = 2$ then we obtain the equation

$$x^q = t_{1,2}(a_{1,2}). \tag{3.10}$$

Make an insertion of indices $\alpha_{1,j} \in \mathbb{Q}$ such that

$$1 < \alpha_{1,1} < \alpha_{1,2} < \cdots < \alpha_{1,q-1} < 2.$$

Consider the group $\mathrm{UT}_{q+1}(\mathbb{F}_p)$, generated by transvections

$$t'_{1,\alpha_{1,1}}, t'_{\alpha_{1,1},\alpha_{1,2}}, \ldots, t'_{\alpha_{1,q-1},2},$$

and define the mapping $\phi_1 : \mathrm{UT}_2(\mathbb{F}_p) \to \mathrm{UT}_{q+1}(\mathbb{F}_p)$ by $\phi_1(t_{1,2}) = t'_{1,2}$. In $\mathrm{UT}_{q+1}(\mathbb{F}_p)$ take the element

$$x_1 = t'_{\alpha_{1,q-1},2} \ldots t'_{\alpha_{1,1},\alpha_{1,2}} t'_{1,\alpha_{1,1}}(a_{1,2}),$$

then $x_1^q = t'_{1,2}(a_{1,2})$. Hence x_1 is a solution of (3.10).

Induction step. Suppose that the equation

$$x^q = a \tag{3.11}$$

over $\mathrm{UT}_{n+1}(\mathbb{F}_p)$ is given. Consider this equation over the factor $\mathrm{UT}_n(\mathbb{F}_p) = \mathrm{UT}_{n+1}(\mathbb{F}_p)/\mathrm{LC}_{n+1}$. By induction the latter is solvable with respect to embedding ϕ of the form (3.6) and its solution x has the form (3.9).

Make an insertion of indices $\alpha_{n,j} \in \mathbb{Q}$ such that

$$n < \alpha_{n,1} < \alpha_{n,2} < \cdots < \alpha_{n,q-1} < n+1.$$

Consider the embedding $\phi' : \mathrm{UT}_{n+1}(\mathbb{F}_p) \to \mathrm{UT}_{nq+1}(\mathbb{F}_p)$ defined by

$$\begin{aligned} \phi'(t_{k,k+1}) &= \phi(t_{k,k+1}), \quad k = 1, \ldots, n-1, \\ \phi'(t_{n,n+1}) &= t'_{n,n+1} \underbrace{t'_{\alpha_{n-1,1},\alpha_{n,1}} \ldots t'_{\alpha_{n-1,q-1},\alpha_{n,q-1}}}_{\Delta_n}. \end{aligned} \tag{3.12}$$

Compute the image of a under ϕ'. It is clear that

$$a = \prod_{i=1}^{n} t_{i,n+1}(a_{i,n+1})\bar{a},$$

where $\bar{a} \in \mathrm{B}_{n+1}$ (in notations of (3.1)). Hence

$$\phi'(a) = \phi'\left(\prod_{i=1}^{n} t_{i,n+1}(a_{i,n+1})\right)\phi(\bar{a}).$$

Observe that $\phi(\bar{a}) = x^q$. Computing for $i = n-1, \dots, 1$ the values $\phi'(t_{i,n+1}) = \phi'([t_{i,i+1}, t_{i+1,n+1}])$ we obtain

$$\begin{aligned} \phi'(t_{1,n+1}) &= t'_{1,n+1} = t'_{1,n+1}\Delta_1, \\ \phi'(t_{i,n+1}) &= t'_{i,n+1}\underbrace{t'_{\alpha_{i-1,1},\alpha_{n,1}} t'_{\alpha_{i-1,2},\alpha_{n,2}} \dots t'_{\alpha_{i-1,q-1},\alpha_{n,q-1}}}_{\Delta_i}, \qquad i = 2, \dots, n-1. \end{aligned}$$

Observe that Δ_i and $t'_{i,n+1}$ commute and also Δ_i and Δ_j commute. Denote $\Delta_i(\gamma) = \Delta_i^{\gamma}$ then $\phi'(t_{i,n+1}(\gamma)) = t'_{i,n+1}(\gamma)\Delta_i(\gamma)$ and

$$\phi'\left(\prod_{i=1}^{n} t_{i,n+1}(a_{i,n+1})\right) = \left(\prod_{i=1}^{n} t'_{i,n+1}(a_{i,n+1})\right)\Delta_2(a_{2,n+1})\dots\Delta_n(a_{n,n+1}).$$

Take

$$\begin{aligned} x_n &= t'_{\alpha_{n,q-1},n+1} \dots t'_{n,\alpha_{n,1}}(a_{n,n+1}) t'_{n-1,\alpha_{n,1}}(a_{n-1,n+1}) \dots t'_{1,\alpha_{n,1}}(a_{1,n+1}) \\ &= e + a_{1,n+1}e_{1,\alpha_{n,1}} + \dots + a_{n,n+1}e_{n,\alpha_{n,1}} + e_{\alpha_{n,1},\alpha_{n,2}} + \dots + e_{\alpha_{n,q-1},n+1} \\ &= e + y, \end{aligned}$$

then $x_n^q = \prod_{i=1}^{n} t'_{i,n+1}(a_{i,n+1})$ and

$$\phi'(a) = x_n^q \Delta_2(a_{2,n+1}) \dots \Delta_n(a_{n,n+1}) x^q.$$

We will show that $x' = x_n x$ is a solution of (3.11) with respect to embedding (3.12). Write $x = e + z$. Since $yz = 0$, we get

$$\begin{aligned} (x_n x)^q &= (e + y + z)^q \\ &= e + y^q + z^q + z^{q-1}y + z^{q-2}y^2 + \dots + zy^{q-1} \\ &= (e + y^q)(e + z^{q-1}y + z^{q-2}y^2 + \dots + zy^{q-1})(e + z^q) \\ &= x_n^q(e + z^{q-1}y + z^{q-2}y^2 + \dots + zy^{q-1})x^q. \end{aligned}$$

Observe that

$$z^{q-k}y^k = a_{2,n+1}e_{\alpha_{1,k},\alpha_{n,k}} + a_{3,n+1}e_{\alpha_{2,k},\alpha_{n,k}} + \dots + a_{n,n+1}e_{\alpha_{n-1,k},\alpha_{n,k}},$$

hence

$$e + z^{q-1}y + z^{q-2}y^2 + \dots + zy^{q-1} = \Delta_2(a_{2,n+1}) \dots \Delta_n(a_{n,n+1}).$$

Finally we obtain $(x_n x)^q = \phi'(a)$, so $x_n x$ is a solution of (3.11) with respect to embedding (3.12).

In a similar way one can prove that solution of (3.8) with respect to embedding (3.7) has the form

$$x = x_1 x_2 \dots x_{n-1},$$

where

$$\begin{aligned}
x_1 &= t'_{\alpha_{n-1,q-1},n}(a_{n-1,n})t'_{\alpha_{n-1,q-2},\alpha_{n-1,q-1}} \dots t'_{n-1,\alpha_{n-1,1}},\\
x_2 &= t'_{\alpha_{n-2,q-1},n}(a_{n-2,n})t'_{\alpha_{n-2,q-1},n-1}(a_{n-2,n-1})t'_{\alpha_{n-2,q-2},\alpha_{n-2,q-1}} \dots t'_{n-2,\alpha_{n-2,1}},\\
&\dots\\
x_{n-1} &= t'_{\alpha_{1,q-1},n}(a_{1,n}) \dots t'_{\alpha_{1,q-1},2}(a_{1,2})t'_{\alpha_{1,q-2},\alpha_{1,q-1}} \dots t'_{1,\alpha_{1,1}}.
\end{aligned}$$

□

Observe that the theorem above performs *simultaneous* adjunction of q-th roots to $\mathrm{UT}_n(\mathbb{F}_p)$, i.e., any element of $\mathrm{UT}_n(\mathbb{F}_p)$ has a q-th root in $\mathrm{UT}_m(\mathbb{F}_p)$.

Example 3.5. Let $n = p = 3$ and $a = \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{UT}_3(\mathbb{F}_3)$. According to Theorem 3.4, solution of $x^3 = a$ over $\mathrm{UT}_3(\mathbb{F}_3)$ with respect to embedding (3.6) has the form

$$x = \begin{pmatrix} 1 & a_{12} & 0 & 0 & a_{13} & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & a_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{UT}_7(\mathbb{F}_3).$$

With respect to embedding (3.7) solution has the form

$$x = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & a_{12} & 0 & 0 & a_{13} \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & a_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{UT}_7(\mathbb{F}_3).$$

4. Embeddings of wreath products

Let $t_{i,j}$ denote a transvection in the group $\mathrm{UT}_n(\mathbb{F}_p)$, where $n \geq 2$, $q = p^s$, $s \in \mathbb{Z}^+$, and let $\alpha_{i,j} \in \mathbb{Q}$ be such that

$$i < \alpha_{i,1} < \cdots < \alpha_{i,q-1} < i+1, \qquad i = 1, \dots, n-1.$$

Let $\mathrm{UT}_m(\mathbb{F}_p)$, where $m = (n-1)q + 1$, be generated by

$$t'_{i,\alpha_{i,1}}, t'_{\alpha_{i,1},\alpha_{i,2}}, \dots, t'_{\alpha_{i,q-1},i+1}, \qquad i = 1, \dots, n-1.$$

Lemma 4.1. *The mapping $\theta : UT_n(\mathbb{F}_p) \to UT_m(\mathbb{F}_p)$, defined by*

$$\theta(t_{i,i+1}) = t'_{i,\alpha_{i,1}} t'^{-1}_{i,\alpha_{i,2}} t'_{i,\alpha_{i,3}} \dots t'^{-1}_{i,\alpha_{i,q-1}} t'_{i,i+1}, \qquad i = 1, \dots, n-1,$$

is an embedding.

Proof. From the following identity

$$[x, yz] = [x, z][x, y][x, y, z]$$

we obtain

$$\theta(t_{i,j}) = t'_{i,\alpha_{j-1,1}} t'^{-1}_{i,\alpha_{j-1,2}} t'_{i,\alpha_{j-1,3}} \dots t'^{-1}_{i,\alpha_{j-1,q-1}} t'_{i,j}.$$

It is easy to check that relations (3.2) hold for $\theta(t_{i,j})$ and $\theta(g) \neq 1$ for $g \neq 1$. □

Lemma 4.2. *Let p be a prime, $q = p^s$, $s \in \mathbb{Z}^+$,*

$$A = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ & 1 & 1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ & & & 1 & 1 \\ 0 & & \dots & & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & -1 & 1 & \dots & -1 & 1 \end{pmatrix} \in M_{q\times q}(\mathbb{F}_p)$$

and $M_i = A^{-i}BA^i$, for $i = 0, \dots, q-1$ ($M_0 = B$). Then the following holds:

1) $(1, -1, 1, \dots, -1, 1)M_i = (0, \dots, 0)$ *for* $i = 1, \dots, q-1$,
2) $\sum_{i=0}^{q-1}(1, -1, 1, \dots, -1, 1)A^i = (0, \dots, 0, 1)$,
3) $\sum_{i=0}^{q-1} M_i = E$.

Proof. In the matrix M_i each column j ($j = 2, \dots, q$) is a multiple of the first one. Indeed, for $M_0 = B$ this statement holds and multiplications by A^{-1} on the left and by A on the right preserve this property. Multiplication by A on the right doesn't change the first column. Hence, to prove the first statement it is enough to show that (notice that $A^{-i} = A^{q-i}$)

$$(1, -1, 1, \dots, -1, 1)A^i \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = 0, \quad \textit{for} \;\; i = 1, \dots, q-1.$$

If $\mathbf{x}^T = (x_q, \dots, x_1)$ then $A\mathbf{x} = (x_q + x_{q-1}, \dots, x_2 + x_1, x_1)$. Further

$$(1, -1, 1, \dots, -1, 1) \begin{pmatrix} x_q + x_{q-1} \\ \vdots \\ x_2 + x_1 \\ x_1 \end{pmatrix} = x_q = 0,$$

since in matrices

$$\begin{pmatrix} y_q \\ \vdots \\ y_2 \\ y_1 \end{pmatrix} = A^i \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad i = 1, \dots, q-1,$$

the value of y_q is equal to 0. This proves the first statement.

The second statement follows from the third one, since multiplication of B by A^{-1} on the left doesn't change the last row of B.

Denote $M = \sum_{i=0}^{q-1} M_i$. Observe that all M_i (and respectively M) are lower triangular matrices and $A^{-1}MA = M$. Then from the system of linear equations $XA = AX$ on the unknown lower triangular matrix X it follows that $X = \lambda E$. Since $M_0 M = M_0$ we have $\lambda = 1$ and $M = E$. □

Theorem 4.3. *The wreath product $UT_n(\mathbb{F}_p) \wr C_q$ $(n \geq 2)$ of unitriangular group $UT_n(\mathbb{F}_p)$ with the cyclic group of order $q = p^s$, $s \in \mathbb{Z}^+$, embeds in $UT_m(\mathbb{F}_p)$, where $m = (n-1)q + 1$.*

Proof. Let $t_{i,j}$ denote a transvection in the group $\mathrm{UT}_n(\mathbb{F}_p)$. Let $\alpha_{i,j} \in \mathbb{Q}$ be such that

$$i < \alpha_{i,1} < \cdots < \alpha_{i,q-1} < i+1, \qquad i = 1, \ldots, n-1,$$

and let $\mathrm{UT}_m(\mathbb{F}_p)$, where $m = (n-1)q+1$, be generated by

$$t'_{i,\alpha_{i,1}}, t'_{\alpha_{i,1},\alpha_{i,2}}, \ldots, t'_{\alpha_{i,q-1},i+1}, \qquad i = 1, \ldots, n-1.$$

By a we denote the generator of C_q. In $\mathrm{UT}_m(\mathbb{F}_p)$ we will construct an element c of order q and subgroups $G_1, \ldots, G_q$ such that the following conditions hold:

1) $G_{i+1} = c^{-1} G_i c$, for $i = 1, \ldots, q-1$;
2) $G_i \simeq \mathrm{UT}_n(\mathbb{F}_p)$, $\phi_i : \mathrm{UT}_n(\mathbb{F}_p) \to G_i$ is a corresponding isomorphism and $\phi_{i+1}(t_{j,j+1}) = c^{-1}\phi_i(t_{j,j+1})c$, for $j = 1, \ldots, n-1$;
3) G_i and G_j are commuting element-wise for $i \neq j$;
4) $G_i \cap G_j = \{1\}$ for $i \neq j$.

Then the mapping $\tau : \mathrm{UT}_n(\mathbb{F}_p) \wr C_q(a) \to \mathrm{UT}_m(\mathbb{F}_p)$, defined by

$$\tau : a^k(h_1, h_2, \ldots, h_q) \mapsto c^k \phi_1(h_1)\phi_2(h_2) \ldots \phi_q(h_q), \tag{4.1}$$

is an embedding. Observe that according to Lemma 2.4 this value of m is the minimal possible.

Denote $g_{i,j} = \phi_i(t_{j,j+1})$, for $i = 1, \ldots, q$, $j = 1, \ldots, n-1$. To prove 3) we will show that $[g_{k,i}, g_{l,j}] = 1$ for $k \neq l$ and $i, j = 1, \ldots, n-1$. Since $g_{k+1,i} = c^{-1} g_{k,i} c$ for $k = 1, \ldots, q-1$ and

$$[g_{k,i}, g_{l,j}] = 1 \Longleftrightarrow [g_{1,i}, c^{-(l-k)} g_{1,j} c^{l-k}] = 1,$$

it is enough to prove that $[g_{1,i}, g_{l,j}] = 1$ for $l = 2, \ldots, q$ and $i, j = 1, \ldots, n-1$.

From 3) it follows that to prove 4) it is enough to show that $\zeta(G_i) \cap \zeta(G_j) = \{1\}$ for $i \neq j$.

For $i = 1, \ldots, n-1$ denote

$$\begin{aligned} c_i &= e + e_{\alpha_{i,1},\alpha_{i,2}} + e_{\alpha_{i,2},\alpha_{i,3}} + \cdots + e_{\alpha_{i,q-1},i+1} \\ &= t'_{\alpha_{i,q-1},i+1} \ldots t'_{\alpha_{i,2},\alpha_{i,3}} t'_{\alpha_{i,1},\alpha_{i,2}} \end{aligned} \tag{4.2}$$

and $c = c_1 c_2 \ldots c_{n-1}$. Clearly $[c_i, c_j] = 1$ and c_i has order q, hence c has order q.

Define the ordered sets

$$\begin{aligned} &I_1 = \{1\}, \\ &I_i = \{\alpha_{i-1,1}, \alpha_{i-1,2}, \ldots, \alpha_{i-1,q-1}, i\}, \qquad i = 2, \ldots, n. \end{aligned}$$

Write

$$h_k = \prod_{\substack{i \in I_k, \\ j \in I_{k+1}}} t'_{i,j}(\gamma_{i,j}), \tag{4.3}$$

where $\gamma_{i,j} \in \mathbb{F}_p$ and $k = 1, \ldots, n-1$. Observe that in the product above all transvections commute. With h_k we associate $|I_k| \times |I_{k+1}|$ matrix $M(h_k) = (\gamma_{i,j})$ over the field $\mathbb{F}_p$, with rows and column indexed by I_k and I_{k+1} respectively. And conversely, with any such matrix we associate an element of the form (4.3). Further we will reduce operations with elements of the form (4.3) to operations with corresponding matrices.

Observe that

$$\begin{aligned} c^{-1}h_1c &= c_1^{-1}h_1c_1, \\ c^{-1}h_kc &= c_{k-1}^{-1}c_k^{-1}h_kc_kc_{k-1}, \quad k = 2, \ldots, n-1. \end{aligned}$$

From (3.2) we obtain

$$\begin{aligned} t'_{i,j}(\alpha)^{t'_{j,k}(\beta)} &= t'_{i,k}(\alpha\beta)t'_{i,j}(\alpha), \quad \alpha, \beta \in \mathbb{F}_p, \\ t'_{j,k}(\beta)^{t'_{i,j}(\alpha)} &= t'_{i,k}(-\alpha\beta)t'_{j,k}(\beta), \\ t'_{i,j}(\alpha)^{t'_{k,l}(\beta)} &= t'_{i,j}(\alpha), \quad j \neq k,\ i \neq l. \end{aligned}$$

Thus $c^{-1}h_kc$ and h_k $(k = 1, \ldots, n-1)$ are elements of the form (4.3). For $k = 1, \ldots, n-1$ we have $M(c_k^{-1}h_kc_k) = M(h_k)A$, where

$$A = \begin{pmatrix} 1 & 1 & 0 & \ldots & 0 \\ & 1 & 1 & \ldots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ & & & 1 & 1 \\ 0 & & \ldots & & 1 \end{pmatrix} \in \mathrm{UT}_q(\mathbb{F}_p).$$

For $k = 2, \ldots, n-1$ we have $M(c_{k-1}^{-1}h_kc_{k-1}) = A^{-1}M(h_k)$. Combining all the above we obtain

$$\begin{aligned} M(c^{-1}h_1c) &= M(h_1)A, \\ M(c^{-1}h_kc) &= A^{-1}M(h_k)A, \qquad k = 2, \ldots, n-1. \end{aligned}$$

Take

$$h_k = t'_{k,\alpha_{k,1}}t'^{-1}_{k,\alpha_{k,2}}t'_{k,\alpha_{k,3}} \ldots t'^{-1}_{k,\alpha_{k,q-1}}t'_{k,k+1}, \tag{4.4}$$

then $M(h_1) = (1, -1, 1, \ldots, -1, 1)$ (for $p = 2$ we treat it as $(1, 1, \ldots, 1)$) and

$$M(h_k) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & -1 & 1 & \ldots & -1 & 1 \end{pmatrix}, \quad k = 2, \ldots, n-1.$$

Take $g_{1,k} = h_k$, for $k = 1, \ldots, n-1$, then by Lemma 4.1 the subgroup $G_1 = \langle g_{1,1}, \ldots, g_{1,n-1} \rangle$ is isomorphic to $\mathrm{UT}_n(\mathbb{F}_p)$. Further take $g_{i+1,k} = c^{-1}g_{i,k}c$, for $i = 1, \ldots, q-1$ and $k = 1, \ldots, n-1$, and $G_i = \langle g_{i,1}, \ldots, g_{i,n-1} \rangle$. This proves statements 1) and 2).

Since $c^{-1}h_kc$ and h_k are elements of the form (4.3) then $[g_{1,k}, g_{l,k}] = 1$, for $k = 1, \ldots, n-1$, $l = 2, \ldots, q$. It is also clear that $[g_{1,k}, g_{l,j}] = 1$ for $|j-k| > 1$. So it remains to consider the case when $|j-k| = 1$. Let $g_{1,k} = h_k = e + \mathcal{A}$

and $g_{l,k+1} = c^{-(l-1)} h_{k+1} c^{l-1} = e + \mathcal{B}$. We will show that $\mathcal{AB} = \mathcal{BA}$. Clearly $\mathcal{BA} = 0$ and

$$\mathcal{AB} = 0 \Longleftrightarrow M(h_k) M(c^{-(l-1)} h_{k+1} c^{l-1}) = 0.$$

The latter follows from the first statement of Lemma 4.2. Thus statement 3) is proved.

Observe that $\zeta(G_1) = \langle z_1 \rangle$, where

$$z_1 = t'_{1,\alpha_{n-1,1}} t'^{-1}_{1,\alpha_{n-1,2}} \dots t'^{-1}_{1,\alpha_{n-1,q-1}} t'_{1,n}.$$

Denote $z_l = c^{-(l-1)} z_1 c^{l-1}$, for $l = 2, \dots, q$. Define y_i by

$$(y_1, \dots, y_q) = (1, -1, \dots, -1, 1) A^{l-1},$$

then

$$z_l = t'^{y_1}_{1,\alpha_{n-1,1}} t'^{y_2}_{1,\alpha_{n-1,2}} \dots t'^{y_{q-1}}_{1,\alpha_{n-1,q-1}} t'^{y_q}_{1,n}.$$

The centers $\zeta(G_l) = \langle z_l \rangle$ are disjoint. This proves statement 4) and brings our proof to the end. $\square$

Let $\rho : \mathrm{UT}_n(\mathbb{F}_p) \to \mathrm{UT}_n(\mathbb{F}_p) \wr C_q$ be the embedding of $\mathrm{UT}_n(\mathbb{F}_p)$ into the diagonal subgroup of the base group, $\tau : \mathrm{UT}_n(\mathbb{F}_p) \wr C_q(c) \to \mathrm{UT}_m(\mathbb{F}_p)$ be embedding (4.1), constructed in Theorem 4.3, and $\phi : \mathrm{UT}_n(\mathbb{F}_p) \to \mathrm{UT}_m(\mathbb{F}_p)$ be embedding (3.6).

Lemma 4.4. *$\tau \circ \rho \equiv \phi$.*

Proof. Using notations of Theorem 4.3 we will compute the diagonal subgroup of the base group. Denote

$$f_k = \prod_{l=1}^{q} g_{l,k}, \qquad k = 1, \dots, n-1,$$

then f_k is an element of the type (4.3). Observe that

$$M(f_1) = \sum_{l=1}^{q} M(g_{l,1}) = \sum_{l=0}^{q-1} M(h_1) A^l,$$
$$M(f_k) = \sum_{l=1}^{q} M(g_{l,k}) = \sum_{l=0}^{q-1} A^{-l} M(h_k) A^l, \quad k = 2, \dots, n-1.$$

From Lemma 4.2 (statements 2) and 3)) it follows that

$$M(f_1) = (0, \dots, 0, 1),$$
$$M(f_k) = E, \qquad k = 2, \dots, n-1.$$

Hence $\phi(t_{i,i+1}) = \tau(\rho(t_{i,i+1}))$, for $i = 1, \dots, n-1$.

$\square$

Open question. Does there exist a nontrivial variety $\mathbb{L}$ of groups, distinct from the variety $\mathbb{G}$ of all groups and the variety $\mathbb{A}$ of all abelian groups, such that every group $G \in \mathbb{L}$ is isomorphic to a subgroup of a divisible group $\bar{G} \in \mathbb{L}$?

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